# Too Much Information on Inclusion-Exclusion

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## 1 Introduction

All the methods we use to compute the surface and volume of a union of spheres are based on this most basic Inclusion-Exclusion formula.

## 2 Basics

A sphere, $S$, is defined as all the points a fixed distance from its center. A ball, $B$, is defined as a sphere with the inside included.

A set of spheres is denoted $\mathbb{S}$ and a set of balls is denoted $\mathbb{B}$. The measure of the object or union of the set of objects is written $|\ \ |$.

## 3 For Balls

Let

$$\mathbb{B} = \left\{B_i \mid i = 1...n\right\}$$

Then

$$\begin{aligned}|\mathbb{B}| \;\; &= \;\; \sum_{i=1}^{n}|B_i| - \sum_{1\leq i_1<i_2\leq n}|B_{i_1}\cap B_{i_2}| + \sum_{1\leq i_1<i_2<i_3\leq n}|B_{i_1}\cap B_{i_2}\cap B_{i_3}| \\ &\quad - \;\; ... \;\; + (1)^{n+1}|B_1\cap B_2\cap ...\cap B_n| \end{aligned} \tag{1}$$

*Proof.* We will use induction.

**Basic Step:**

$$|\{B_1, B_2\}| = \sum_{i=1}^{2}|B_i| - |B_1\cap B_2| \tag{2}$$

This is obvious and it applies to all sets, especially bounded ones. Just look at a picture.

**Inductive Step:**

We will use

$$|\mathbb{B}| = \sum_{i=1}^{n}|B_i| - \sum_{1\leq i_1<i_2\leq n}|B_{i_1}\cap B_{i_2}| + \;\; ... \;\; + (1)^{n+1}|B_1\cap B_2\cap ...\cap B_n| \tag{3}$$

to show

$$\left|\mathbb{B}\cup\{B_{n+1}\}\right| = \sum_{i=1}^{n+1}|B_i| - \sum_{1\leq i_1<i_2\leq n+1}|B_{i_1}\cap B_{i_2}| + \;\; ... \;\; + (1)^{n+2}|B_1\cap B_2\cap ...\cap B_n\cap B_{n+1}| \tag{4}$$

$$
\begin{aligned}
& \left| \bigcup_{i=1}^{n+1} |B_i| \right| \\
= \; & \left| \left( \bigcup_{i=1}^{n} B_i \right) \cup B_{n+1} \right| \\
= \; & \left| \bigcup_{i=1}^{n} B_i \right| + |B_{n+1}| - \left| \left( \bigcup_{i=1}^{n} B_i \right) \cap B_{n+1} \right| \qquad \text{by (2)} \\
= \; & \left| \bigcup_{i=1}^{n} B_i \right| + |B_{n+1}| - \left| \bigcup_{i=1}^{n} (B_i \cap B_{n+1}) \right| \qquad \text{by an elementary properties of sets} \qquad (5) \\
= \; & \sum_{i=1}^{n} B_i - \sum_{1 \le i_1 < i_2 \le n} |B_{i_1} \cap B_{i_2}| + \sum_{1 \le i_1 < i_2 < i_3 \le n} |B_{i_1} \cap B_{i_2} \cap B_{i_3}| \\
& - \; ... \; + (1)^{n+1} |B_1 \cap B_2 \cap ... \cap B_n| \\
& + |B_{n+1}| \\
& \sum_{i=1}^{n} |B_i \cap B_{n+1}| - \sum_{1 \le i_1 < i_2 \le n} |B_{i_1} \cap B_{i_2} \cap B_{n+1}| + \sum_{1 \le i_1 < i_2 < i_3 \le n} |B_{i_1} \cap B_{i_2} \cap B_{i_3} \cap B_{n+1}| \\
& - \; ... \; + (1)^{n+2} |B_1 \cap B_2 \cap ... \cap B_n \cap B_{n+1}| \qquad (6)
\end{aligned}
$$

Now just match up the parts and you have (4). □

# 4 For Spheres

The surface of the union of $n$ spheres is the surface of the union of the balls induced by those spheres. The same applies for intersections. The surface of the intersection of $n$ spheres is the surface of the intersection of the balls induced by those spheres.

The proof for spheres is similar to the one for balls. The only questionable part is (5).

$$\left( \bigcup_{i=1}^{n} S_i \right) \cap S_{n+1} = \bigcup_{i=1}^{n} \left( S_i \cap S_{n+1} \right) \tag{7}$$

*Proof.* Again, we will use induction.

**Basic Step:**

For $n = 1$ the above is trivial.

**Inductive Step:**

We will use

$$\left( \bigcup_{i=1}^{n} S_i \right) \cap S_{n+1} = \bigcup_{i=1}^{n} \left( S_i \cap S_{n+1} \right) \tag{8}$$

to show

$$\left( \bigcup_{i=1}^{n+1} S_i \right) \cap S_{n+2} = \bigcup_{i=1}^{n+1} \left( S_i \cap S_{n+2} \right) \tag{9}$$

$$
\begin{aligned}
& \left(\bigcup_{i=1}^{n+1} S_i\right) \cap S_{n+2} \\
= \quad & \left(\left(\bigcup_{i=1}^{n} S_i\right) \cup S_{n+1}\right) \cap S_{n+2} \\
= \quad & \left(\left(\bigcup_{i=1}^{n} S_i\right) \cap S_{n+2}\right) \cup \big(S_{n+1} \cap S_{n+2}\big) && \text{by the basic step} \\
= \quad & \left(\bigcup_{i=1}^{n} \big(S_i \cap S_{n+2}\big)\right) \cup \big(S_{n+1} \cap S_{n+2}\big) && \text{by (8)} \\
= \quad & \bigcup_{i=1}^{n+1} \big(S_i \cap S_{n+2}\big)
\end{aligned}
$$

□

With this results the proof for balls can be used as a template. One just needs to replace every "B" with a "S."